Bruhat Decomposition is Linear Algebra

Riley Moriss

August 13, 2025

This is a nice application of first year linear algebra. We will prove that for GL_n we have the decoposition $G = BWB$, where B is the Borel of upper triangular matrices and W is the Weyl group, in this case isomorphic to S_n (the symmtric group).

First consider the torus of diagonal matrices T. Then its normalizer clearly contains both T and the elementary row operations given by swapping the i-th and $i+1$-th row, which for say GL_4 are given by

$$\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$$

because if we consider ntn^{-1} where t is diagonal then you swap the two rows and then just swap them back. Moreover there is a clear bijection between S_n and these matrices given by sending the elementary permutation $(i \quad i+1)$ to the matrix that swaps the i-th and $i+1$-th row as above. Hence it is clear that these matrices actually are the Weyl group (the normalising elements modulo those that are in the torus).

Now the other elementary row operations are adding a multiple of one row to another (or itself). If we want to say put our matrix into upper triangular form then these matricies will be given by multiplying on the left by *lower triangular* matrices. For instance adding x times the first row (the column of x) to the third row (the row of x) will be given by

$$\begin{pmatrix} 1 & & \\ & 1 & \\ x & & 1 \end{pmatrix}$$

Thus all the elementary row operations are either lower triangular or in the Weyl group! Thus executing Gaussian elimination, putting the matrix into upper triangular form writes a matrix as

$$A = L w_1 \cdots w_k U$$

for a lower triangular matrix L (the product of our operations adding rows to others), a product of elements in $w_i \in W$ (the row permutation steps) and an upper triangular matrix U (the row echelon form of the matrix). By adding some more elements $w_l \cdots w_{-1}$ in front of our Weyl group elements we can permute the rows of the lower triangular matrix to make it *upper triangular*, thus putting our matrix in the form

$$A = U_1 w_1 \cdots w_{k'} U_2$$

in other words into the set

$$BWB.$$